TENNESSEE MATHEMATICS TEACHERS ASSOCIATION

SIXTY-SIXTH ANNUAL MATHEMATICS CONTEST

2024

Calculus

Prepared by:

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Scoring Formula: $4 \times$ (Number Right) $-$ (Number Wrong) $+ 40$

Directions:

Do not open this booklet until you are told to do so.

This is a test of your competence in high school mathematics. For each problem, determine the best answer and indicate your choice by making a heavy black mark in the proper place on the separate answer sheet provided. You must use a pencil with a soft lead (No. 2 lead or softer).

This test has been constructed so that most of you are not expected to answer all of the questions. Do your best on the questions you feel you know how to work. You will be penalized for incorrect answers, so wild guesses are not advisable.

If you change your mind about an answer, be sure to erase completely. Do not mark more than one answer for any problem. Make no stray marks of any kind on the answer sheet. The answer sheets will not be returned to you; if you wish a record of your performance, mark your answers in this booklet also. You will keep the booklet after the test is completed.

When told to do so, open your test booklet and begin. You will have exactly eighty minutes to work.

1. What type of conic section is described by the equation $9x^2 - 6x - 16y^2 + 24y + 10 = 0$?

   a. Circle b. Ellipse c. Hyperbola d. Parabola e. Point

2. What is $\frac{3-2i}{2+3i}$ written in standard form?

   a. 1 b. $\frac{12}{13}+i$ c. $\frac{12}{13}-i$ d. i e. $-i$

3. What is the average rate of change for the function $f(x) = x^3 - 3x^2$ on the interval from $x = 1$ to $x = 3$?

   a. -2 b. -1 c. 0 d. 1 e. 2

4. In Triangle ABC, $m\overline{AB} = 4$ cm, $m\angle ACB = 40°$, and $m\angle BAC = 60°$. What is $m\overline{BC}$?

   a. 6.13 cm b. 6.00 cm c. 5.93 cm d. 5.39 cm e. 2.97 cm

5. If $\sec\theta = 3$ and θ is in quadrant IV, what is the value of $\cot\theta$?

   a. $2\sqrt{2}$ b. $\frac{\sqrt{2}}{4}$ c. $\frac{1}{3}$ d. $-2\sqrt{2}$ e. $-\frac{\sqrt{2}}{4}$

6. What is the equation of the tangent line to the curve $f(x) = 3x - 4x^2$ at $x = -1$?

   a. $y = -5x - 1$ b. $y = -5x - 7$ c. $y = -5x - 8$ d. $y = 11x + 4$ e. $y = 11x - 7$

7. What is the derivative of $\sin\left(3x^2\right)$?

   a. $\cos\left(3x^2\right)$ b. $\cos(6x)$ c. $6x\sin\left(3x^2\right)$ d. $-6x\cos\left(3x^2\right)$ e. $6x\cos\left(3x^2\right)$

8. Where is the curve $f(x) = x^2 - 4\cos x$ concave down in the interval $[0, 2\pi]$?

   a. $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$ b. $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$ c. $\left[0, \frac{2\pi}{3}\right) \cup \left(\frac{4\pi}{3}, 2\pi\right]$ d. $\left[0, \frac{2\pi}{3}\right] \cup \left[\frac{4\pi}{3}, 2\pi\right]$ e. $[0, 2\pi]$

9. At how many points is the following function discontinuous?

$$f(x)=\begin{cases}\dfrac{1}{x^2+4x+3} & x<2\\ e^{x-2} & x\geq 2\end{cases}$$

a. 0 b. 1 c. 2 d. 3 e. 4

10. Which of the following statements are true?

I. If $\lim_{x\to 2} f(x)$ exists, then the function is continuous at $x=2$.

II. If $f(x)$ is continuous at $x=2$, then it is differentiable at $x=2$.

III. If $f(x)$ is differentiable at $x=2$, then it is continuous at $x=2$.

a. I only b. II only c. I and II d. III only e. I and III

11. What are the critical points of the function $f(x)=|x-2|(x+1)$?

a. $\frac{1}{2}$ b. $-1,2$ c. $-2,1$ d. $\frac{1}{2},2$ e. $-1,\frac{1}{2},2$

12. What is the absolute maximum value of the function $f(x)=|x-2|(x+1)$ on the interval [0,3]?

a. $\frac{9}{2}$ b. 4 c. $\frac{9}{4}$ d. 2 e. 0

13. What is the derivative of $f(x)=\frac{1+x^4}{x^2}$?

a. $\frac{x^5-2}{x^4}$ b. $\frac{x^4-2}{x^3}$ c. $\frac{2x^4-2}{x^3}$ d. $4x^2$ e. $2x$

14. What is the slope of the tangent line to the curve $f(x)=x^2\sin x$ when $x=-\frac{\pi}{2}$?

a. $-\frac{\pi^2}{4}$ b. $-\pi$ c. π d. 2π e. $\frac{\pi^2}{4}$

15. What is $\lim_{x\to 3^-} \dfrac{2x^2-5x-3}{3x^2-10x+3}$?

a. 0 b. $\frac{2}{3}$ c. $\frac{7}{8}$ d. 1 e. undefined

16. Using the tangent line for $f(x) = \sqrt[5]{x}$ at $x = 32$, what would be the estimated value of $\sqrt[5]{30}$?

a. 1.976 b. 1.975 c. 1.974 d. 1.973 e. 1.972

17. Suppose that $f(x)$ is differentiable on (0,3) and continuous on [0,3], and we know that $f(0) = -2$ and $f(3) = 3$. Which of the following are true?

I. There exists *c* in (0,3) where $f(c) = 0$
II. There exists *c* in (0,3) where $f'(c) = 0$
III. There exists *c* in (0,3) where $f'(c) = \frac{5}{3}$

a. I only b. II only c. I and II d. I and III e. All three

18. What must the value of a be for the vectors $\langle 1, 2, -1\rangle$ and $\langle 3, a, 5\rangle$ to be perpendicular?

a. 1 b. 2 c. 3 d. 4 e. 5

19. Using the table on the right, what is the derivative of $f^{-1}(x)$ at $x = 4$?

x	$f(x)$	$f'(x)$
3	4	3
4	5	6
5	3	7

a. $\frac{1}{6}$ b. $\frac{1}{3}$ c. 3 d. 5 e. 6

20. What is $f(x)$ if $f''(x) = 3x$, $f'(1) = -\frac{1}{2}$, and $f(1) = \frac{1}{2}$?

a. $\frac{x^3}{2} - 2x + 2$ b. $\frac{x^3}{2} + 2x - 2$ c. $\frac{x^3 - x + 1}{2}$ d. $\frac{x^3 + x - 1}{2}$ e. $\frac{x^3}{2}$

21. What is the value of $\sqrt{3-2\sqrt{3-2\sqrt{3-2\sqrt{\ldots}}}}$?

a. 3 b. 1 c. -1 d. -3 e. 0

22. What is the slope of the curve $x^3-3xy^2+y^3=1$ at the point (2,−1)?

a. -1 b. $-\frac{3}{5}$ c. 1 d. $\frac{6}{5}$ e. $\frac{9}{5}$

23. If we revolve the region bounded by the curves $y=x^2$ and $y=x+2$ about the x-axis, what is the volume of the resulting solid?

a. $\frac{68}{5}\pi$ b. 14π c. $\frac{71}{5}\pi$ d. $\frac{72}{5}\pi$ e. $\frac{73}{5}\pi$

24. What is the value of $\lim_{x\to 0}\frac{1+x-e^x}{x(e^x-1)}$?

a. $-\frac{1}{2}$ b. 0 c. $\frac{1}{2}$ d. 1 e. undefined

25. Using the trapezoidal rule with 3 trapezoids, what is the estimated value of $\int_0^3 2\sqrt{x}\ dx$?

a. $2\sqrt{2}+\sqrt{3}$ b. $2+\sqrt{2}+\sqrt{3}$ c. $2+2\sqrt{2}+\sqrt{3}$ d. $2+2\sqrt{2}+2\sqrt{3}$ e. $4\sqrt{2}+2\sqrt{3}$

26. What is the value of the following sum?

$$\sum_{k=1}^{26}(18+2k)$$

a. 1100 b. 1118 c. 1170 d. 1242 e. 1294

27. What is $\int xe^x dx$?

a. $e^x(x+1)+c$ b. $e^x(x-1)+c$ c. xe^x+c d. $\frac{x^2}{2}e^x+c$ e. $e^x\left(\frac{x^2}{2}-1\right)+c$

28. What is the value of the 99th derivative of $f(x) = x\cos x$ at $x = 0$?

a. -100 b. -99 c. 0 d. 99 e. 100

29. What is $\int \tan(3x)dx$?

a. $\frac{\ln|\sin 3x|}{3} + c$ b. $\frac{\ln|\cos 3x|}{3} + c$ c. $\frac{\ln|\sec 3x|}{3} + c$ d. $\frac{\ln|\tan 3x|}{3} + c$ e. $\frac{\ln|\cot 3x|}{3} + c$

30. What is the area of the region in the first quadrant enclosed by the graphs of $y = 4x^2$ and $y = \sin(\pi x)$?

a. $\frac{1}{\pi} - \frac{1}{6}$ b. $\frac{2}{\pi} - \frac{1}{6}$ c. $\frac{2}{\pi} - \frac{1}{12}$ d. $\frac{1}{\pi} + \frac{1}{6}$ e. $\frac{1}{\pi} + \frac{1}{12}$

31. A conical paper cup with a diameter of 8 inches and 6 inches deep and is full of water. If it is leaking water at 2 cubic inches per minute through the bottom, how fast is the height of the water decreasing, in inches per minute, when the water is exactly 3 inches deep?

a. $\frac{1}{8\pi}$ b. $\frac{1}{4\pi}$ c. $\frac{1}{2\pi}$ d. $\frac{1}{\pi}$ e. $\frac{2}{\pi}$

32. Which of the following is $\int \sin^3 x \cos^3 x\, dx$?

a. $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$

b. $\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} + c$

c. $\frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + c$

d. $\frac{\cos^4 x}{4} - \frac{\cos^6 x}{6} + c$

e. $\frac{\sin^4 x \cos^4 x}{16} + c$

33. What is the value of the following infinite sum?

$$\sum_{k=1}^{\infty} \frac{1}{k(k+3)}$$

a. $\frac{4}{9}$ b. $\frac{1}{2}$ c. $\frac{5}{9}$ d. $\frac{11}{18}$ e. $\frac{2}{3}$

34. What is $\frac{d}{dx}\int_0^{x^3} \frac{dt}{1+t}$?

a. $3x^2 \ln\left|1+x^3\right|$ b. $\ln\left|1+x^3\right|$ c. $\frac{1}{1+x^3}$ d. $\frac{x^3}{1+x^3}$ e. $\frac{3x^2}{1+x^3}$

35. What is the area of the region bounded by the polar equations $r = 2\cos\theta, r = \cos\theta, 0 \le \theta \le \frac{\pi}{4}$?

a. $\frac{3}{4}+\frac{3\pi}{8}$ b. $\frac{3}{4}+\frac{3\pi}{16}$ c. $\frac{3}{4}+\frac{3\pi}{32}$ d. $\frac{3}{8}+\frac{3\pi}{16}$ e. $\frac{3}{8}+\frac{3\pi}{32}$

36. The base of a solid is the region between the parabolas $x = y^2$ and $x = 3-2y^2$. What is the volume of the solid if the cross sections perpendicular to the x-axis are squares?

a. 4 b. 4.5 c. 5 d. 5.5 e. 6

37. What are the first two nonzero terms of Maclaurin series for $f(x) = x\sin x$?

a. $x^2 - \frac{x^4}{3!}$ b. $\frac{x^2}{2!} - \frac{x^4}{4!}$ c. $x - \frac{x^3}{3!}$ d. $x - \frac{x^3}{2!}$ e. $1 - \frac{x^2}{2!}$

38. What is the distance traveled from $t = 0$ to $t = \pi$ by a particle traveling on the parameterized curve $x(t) = e^t \sin t,\ y(t) = e^t \cos t$?

a. $\left(e^{\pi}-1\right)\sqrt{2}$ b. $e^{\pi}\sqrt{2}$ c. $\left(e^{2\pi}-1\right)\sqrt{2}$ d. $e^{2\pi}\sqrt{2}$ e. $e^{\pi^2}\sqrt{2}$

39. Let R be the region below the graph of $f(x)=\frac{1}{x^2}$ where $x\geq 1$. What is the volume of the solid formed by revolving R about the x-axis?

a. π b. $\frac{\pi}{2}$ c. $\frac{\pi}{3}$ d. $\frac{\pi}{4}$ e. undefined

40. If $\int_0^1 \frac{x-3}{(x+1)(x+3)}dx=\ln\left(\frac{A}{B}\right)$ where $\frac{A}{B}$ is a reduced fraction, what is the value of $A+B$?

a. 43 b. 44 c. 45 d. 46 e. 47